Exam Quantum Field Theory
January 21, 2019
Start: 14:00h End: 17:00h

Each sheet <u>with your name and student ID</u>

INSTRUCTIONS: This is a closed-book and closed-notes exam. You are allowed to bring one A4 page written by you on one side, with useful formulas. The exam duration is 3 hours. There is a total of 9 points that you can collect.

NOTE: If you are not asked to **Show your work**, then an answer is sufficient. However, you might always earn more points by answering more extensively (but you can also lose points by adding wrong explanations). If you are asked to **Show your work**, then you should explain your reasoning and the mathematical steps of your derivation in full. Use the official exam paper for *all* your work and ask for more if you need.

USEFUL FORMULAS

For the energy projectors for spin 1/2 Dirac fermions use the normalization <u>without</u> the factor $1/(2m)$:

$$\sum_{r=1,2} u_r(\vec{p})\bar{u}_r(\vec{p}) = \not{p} + m$$

$$\sum_{r=1,2} v_r(\vec{p})\bar{v}_r(\vec{p}) = \not{p} - m$$

$$\{\gamma_5, \gamma^\mu\} = 0, \quad \left(\gamma^0\right)^2 = \mathbb{1}, \quad \gamma_5^2 = \mathbb{1}, \quad \gamma_5^\dagger = \gamma_5, \quad \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma})$$

$$\gamma^\mu\gamma_\mu = 4\mathbb{1} \quad \gamma^\mu\not{p}\gamma_\mu = -2\not{p} \quad \not{k}\not{p}\not{k} = 2(pk)\not{k} - k^2\not{p}$$

$$\mathrm{Tr}(\gamma_5\gamma^\mu) = \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho) = 0$$

1. (3 points total) The Lagrangian density of Scalar QED describes the interactions of a complex scalar field with the electromagnetic field:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^\dagger D^\mu\phi - m^2\phi^\dagger\phi\,,$$

with covariant derivatives

$$\begin{aligned} D_\mu\phi &= \partial_\mu\phi + ieA_\mu\phi \\ D_\mu\phi^\dagger &= \partial_\mu\phi^\dagger - ieA_\mu\phi^\dagger\,, \end{aligned}$$

where A_μ is the electromagnetic field and e the electric charge.

a) **[1 points]** Write explicitly the free Lagrangian density $\mathcal{L}_0$ and the interaction Lagrangian density $\mathcal{L}_I$ for this theory. Draw the Feynman diagrams corresponding to the interaction vertices, without deriving the corresponding Feynman rules.

b) **[2 points]** Write the path integral for this theory, $Z[J_\mu, J, J^\dagger]$, in presence of the external sources, J_μ, J, and $J^\dagger$. Work out its perturbative expansion in powers of the coupling e, or equivalently in powers of the interaction Lagrangian $\mathcal{L}_I$, where the latter is a functional of the derivatives w.r.t. the external sources acting on the path integral of the free theory, $Z_0[J_\mu, J, J^\dagger]$. **Show your work**

2. (2 points total) Consider the $N = 2$ linear σ (sigma) model coupled to fermions:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_i)^2 - \frac{1}{2}m^2\phi_i^2 - \frac{\lambda}{4}(\phi_i^2)^2 + \bar{\psi}i\partial\!\!\!/\psi - g\bar{\psi}(\phi_1 + i\gamma_5\phi_2)\psi\,,$$

with $i = 1, 2$, $\phi_i^2 = \phi_1^2 + \phi_2^2$ and couplings λ and g. In this theory the scalars have mass m and the fermions are massless. Show that this theory has the following global symmetry:

$$\begin{aligned} \phi_1' &= \cos\alpha\,\phi_1 - \sin\alpha\,\phi_2 \\ \phi_2' &= \sin\alpha\,\phi_1 + \cos\alpha\,\phi_2 \\ \psi' &= e^{-i\alpha\gamma_5/2}\psi\,, \end{aligned}$$

and the transformation of $\bar{\psi}$ follows from the one of ψ. **Show your work**

Hint:

$$\begin{aligned} \cos\alpha &= 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \ldots \\ \sin\alpha &= \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \ldots \end{aligned}$$

3. (1.5 points total) Consider the QED Lagrangian density with massive fermions:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(iD\!\!\!\!/ - m\right)\psi\,,$$

with covariant derivative $D_\mu = \partial_\mu + ieA_\mu$. Show that the axial current $J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ is not conserved and its nonconservation is due to the fermion mass. Specifically, show that

$$\partial_\mu J_5^\mu = 2im\bar{\psi}\gamma_5\psi$$

holds on the equations of motion for ψ and $\bar{\psi}$

$$\begin{aligned}(i\overrightarrow{\partial\!\!\!/} - eA\!\!\!/ - m)\psi &= 0\\ \bar{\psi}(i\overleftarrow{\partial\!\!\!/} + eA\!\!\!/ + m) &= 0\end{aligned}$$

Hint: $\{\gamma^\mu, \gamma_5\} = 0$.

4. (2.5 points total) Consider the two-body decay $Z^0 \to \nu\bar{\nu}$ mediated by weak interactions, where Z^0 is an electrically neutral spin-1 boson with mass M, ν is a massless neutrino and $\bar{\nu}$ is a massless antineutrino.
The Feynman rule for the interaction vertex of a Z^0 with $\nu\bar{\nu}$ is $ig_w\gamma^\mu(1-\gamma_5)$, with g_w the weak coupling and μ the Lorentz index of the polarization vector of the external Z^0.
Calculate the decay rate $\Gamma(Z^0 \to \nu\bar{\nu})$, using the general formula for the decay rate of a particle at rest with mass M into two massless particles:

$$\Gamma = \frac{1}{16\pi}\frac{1}{M}X$$

with $X = \left(\mathcal{A}^\dagger\mathcal{A}\right)_{\text{unpol.}}$ the unpolarized squared amplitude, and the sum over fermion spins given on page 1 (i.e. without the normalization factor $1/(2m)$). **Show your work**

Hints:

- For what concerns this exercise, a neutrino ν (antineutrino $\bar{\nu}$) is fully analogous to a massless electron e^- (massless positron e^+) except for the fact that neutrinos and antineutrinos have zero electric charge.
- Use for the sum over the polarizations of the massive vector particle Z^0

  $$\sum_{a=1}^{3}\epsilon_a^\mu(k)\epsilon_a^\nu(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{M^2}$$

  with ϵ_μ real. This is not QED, $k^\mu k^\nu$ terms matter!
- $(\bar{u}\Gamma v)^\dagger = \bar{v}\gamma^0\Gamma^\dagger\gamma^0 u$, with Γ any product of γ matrices.
- You should find that X can be written in terms of the mass M only.